Small-Angle Approximation for the Impedance due to Wall Surface Roughness

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1 INTRODUCTION

The design of Linac Coherent Light Source (LCLS) at SLAC requires the energy spread of the beam to be less then 0.1% [1]. Longitudinal wakefields in the accelerator tend to redistribute the bunch energy and, if large enough, can cause degradation of the beam performance. It has been pointed out by Bane, Chao and Ng [2] , that one of the major sources of wakefields for the LCLS might be the wall surface roughness in the undulator. The model developed in Ref. [2] assumes that a rough surface can be represented as a collection of bumps of relatively simple shapes (hemisheres, half cubes, etc.), and the impedance is calculated a sum of impedances for each shape with account of density of the bumps per unit area.

In this paper, we develop a theory of impedance due to the surface roughness of a perfect conductor, using a small-angle approximation for the wall surface. Our final result is the expression for the impedance in terms of the spectral function of the surface profile. The formula represents the contribution of different scales, and can be used for estimation of the impedance based on statistical properties of the surface. A simple model of a fractal random surface is considered, and it is shown that the impedance is proportional to the rms square of the bumps height divided by the correlation length of the bumps. The result is compared to the Bane-Chao-Ng model.

2 ASSUMPTIONS AND STATISTICAL PROPERTIES OF A ROUGH SURFACE

The detailed derivation of the impedance of round pipe with rough surface can be found elsewhere [3]. Here we outline the main assumptions and present the final formula for the longitudinal impedance.

Our approach is based on the assumption of *small-angle approximation*. If we assume that the rough surface is given by equation $y = h(x, z)$, where x, y and z are the cartesian coordinates, and h is the local height of the surface, then the small-angle approximation means that the angle between the normal to the surface and the vertical direction (y axes) is much smaller than unity, or

$$|\nabla h| \ll 1\,. \tag{1}$$

This assumption allows to develop a rather general theory of impedance, which with a good accuracy works even when ($|\nabla h| \sim 1$).

In addition to Eq. (1), we also require that the height of the bumps and their characteristic size g be small compared to the radius of the pipe b_0,

$$g,\, |h| \ll b_0\,. \tag{2}$$

Evidently, this inequality is easily satisfied for realistic values of g, h and b_0.

There is one more condition that simplifies the consideration. Typically, the size of the surface bumps g is on the order of microns, and the bunch length σ_z in FEL is of the order of at least tens of microns. This means that the characteristic frequency of interest $\omega \sim c/\sigma_z$ is small compared to c/g,

$$\omega \ll c/g\,. \tag{3}$$

To describe the statistical properties of a rough surface, we introduce the correlation function $K(x, y)$ such that

$$K(x - x', z - z') = \langle h(x', z') h(x, z)\rangle\,, \tag{4}$$

where the angular brackets denote averaging over possible random profiles $h(x, z)$. Eq. (4) assumes that statistical properties of the surface do not depend on the position, which is true due to the macroscopic nature of the roughness. An important statistical characteristic of the roughness is the *spectral density* (or *spectrum*) $R(\kappa_z, \kappa_x)$, defined as a Fourier transform of the correlation function,

$$R(\kappa_x, \kappa_z) = \frac{1}{(2\pi)^2} \int dx\, dz\, K(x, z) e^{-i\kappa_x x - i\kappa_z z}\,. \tag{5}$$

If the surface is statistically isotropic (all direction in the $x - y$ plane are statistically equivalent), the spectrum R depends only on the absolute value of the wave number $\kappa = \sqrt{\kappa_x^2 + \kappa_z^2}, R = R(\kappa)$.

The main result of Ref. [3] is that the longitudinal impedance of a circular pipe of radius b_0 with a rough perfectly conducting surface characterized by the spectral function $R(\kappa_x, \kappa_z)$ in the frequency range limited by the condition (3) is given by the following equation:

$$Z(\omega) = -\frac{ikZ_0}{2\pi b_0} \int d\kappa_z\, d\kappa_x\, R(\kappa_x, \kappa_z) \frac{\kappa_z^2}{\kappa}\,, \tag{6}$$

where now the z-axes is directed along the pipe axes, and the x axes is locally directed along the azimuthal coordinate θ. We note again, that due to assumed smallness of the surface structures, we can use the local Cartesian coordinate system x, y and z in Eqs. (4) – (6) instead of the global cylindrical coordinate system θ, r and z.

Eq. (6) tells that the contribution to Z of roughness in longitudinal (z) and azimuthal directions are not equal: the presence of the factor κ_z^2 in the integrand means that bellow-type variations on the surface are more dangerous than ridges on the surface going in the longitudinal direction.

3 SURFACE MODELS

As a model of a rough surface, we consider here a power spectrum, limited at small wavelengths, $R(\kappa) = A\kappa^{-q}$ for $\kappa > \kappa_0$, and $R(\kappa) = 0$ for $\kappa < \kappa_0$, where κ_0 is the minimal value of spectrum, $q > 0$ is a power factor, and A defines the amplitude of the roughness. The parameter κ_0 can be expressed in terms of the characteristic correlation length, l_c, of the random profile, $\kappa_0 \sim \pi/l_c$. We can also relate the factor A to the rms height d of the roughness, using the relation

$$d^2 = 2\pi \int_0^\infty \kappa\, d\kappa\, R(\kappa) = \frac{2\pi A}{q-2}\kappa_0^{2-q} . \qquad (7)$$

For the convergence of the integral we have to require $q > 2$. The shape of the surface for two different values of q

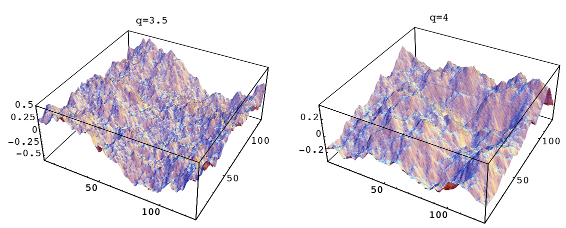

Figure 1: Fractal surfaces for $q = 3.5$ and $q = 4$. Smaller values of q give more "spiky" profiles.

obtained with a help of computer code described in [4] is shown in Fig. 1. It turns out, that increasing the value of q makes the surface smoother. Using Eq. (6) we can calculated the impedance of such surface,

$$Z(\omega) = -\frac{ikZ_0}{4\pi b_0}\frac{q-2}{q-3}\, d^2\kappa_0 . \qquad (8)$$

Again, for convergence, we need to require that $q > 3$, otherwise the integral diverges as $\kappa \to \infty$. This requirement is stronger than the convergence condition for Eq. (7), and is due to a relatively slow decay of the spectrum at large κ.

4 COMPARISON WITH BANE-NG-CHAO MODEL

To compare our result with Ref. [1], we write down here the impedance from [1]

$$Z(\omega) = -\frac{ikf\alpha Z_0}{3\pi^2 b_0}\, a , \qquad (9)$$

where a is the height of the bumps, f is a form-factor that depends on the choice of specific shape modeling the bumps (f varies from about 5 to 20 for different shapes [2]), and α is the filling factor characterizing the number of bumps per unit area. For numerical estimate in [1] it was assumed that $f = 5$ and $\alpha = 0.5$. To compare this result to our model, we have to express the rms height d in terms of a; a simple calculation gives $d = \sqrt{\alpha/2}d$.

Returning to the fractal model, we will chose $q = 4$ as reasonable approximation for a real surface profile. A choice of the correlation length l_c that would be compatible with [1] requires that l_c be of the order of the bump's height, that is $\kappa_0 \sim \pi/d$. This reduces Eq. (8) to

$$Z(\omega) = -\frac{ikZ_0}{2b_0}\, d . \qquad (10)$$

Comparing Eqs. (9) and (10), we see that they have same scaling, but different numerical factors. Eq. (10) gives about three times larger impedance (for $\kappa_0 \sim \pi/d$) than quoted in [1].

We want to emphasize here that the right choice of the correlation length l_c is critical for the estimation of the impedance (8). Although we do not have detailed data for the roughness spectrum of a real surface (which, of course, depends on the particular manufacturing process involved), there are some indication, that in many cases l_c may be 10-30 times larger than assumed above [5]. If that assumption is correct, then the impedance would be about an order of magnitude smaller then estimated above.

5 CONCLUSIONS

We have developed a theory of impedance of perfectly conducting rough surface in small-angle approximation. The effect of finite conductivity is independent of the geometrical wake, and is additive to the one found in this paper. Using as an example a statistically fractal surface with a power spectrum, we calculated the longitudinal impedance as a function of statistical characteristics of the surface.

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7 REFERENCES

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