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Effect of Surface Roughness on Eddy Current Losses at Microwave Frequencies

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A theoretical investigation has been made of the power dissipation by eddy currents in a metallic surface at microwave frequencies in the presence of regular parallel grooves or scratches whose dimensions are comparable to the eddy current skin depth. The eddy current equation has been integrated numerically for grooves of various shapes and sizes transverse to the direction of induced current flow, and the corresponding losses are calculated and plotted. The power dissipation is increased by about 60 percent over its value for a smooth surface when the root-mean-square deviation of the grooved surface from an average plane is equal to the skin depth; the exact shape of the grooves is not critical. The increase in eddy current losses caused by grooves parallel to the current is shown in a particular case to be only about one-third as great as the increase caused by transverse grooves of similar size. The effect on losses of an isolated narrow crack or fissure transverse to the current is briefly discussed.

I. INTRODUCTION

IT is well known that electromagnetic oscillations in a wave guide or cavity resonator are always accompanied by heat losses caused by the flow of induced currents in the metal walls of the structure. The usual formulas for these losses are derived on the assumption of perfectly smooth metal surfaces of conductivity equal to that of the bulk metal. In experimental studies at the Bell Telephone Laboratories, however, it is being found that for wave-lengths in the neighborhood of 3 cm the losses are materially higher than calculated, by amounts ranging roughly from 10 percent to 60 percent. Other work[1] at 1.25 cm has revealed losses similarly in excess of the theoretical values, and the discrepancies tend to become worse at shorter wave-lengths. These discrepancies cannot be explained as attributable to the deviations from Ohm's law which occur[2,3] when the eddy current skin depth becomes less than the mean free path of electrons in the metal, or when the period of the electromagnetic oscillations becomes comparable with the mean time between collisions of an electron with the crystalline lattice, since it now appears that the latter effects will be encountered in good conductors at room temperature only if the wave-length is shorter than a few tenths of a millimeter. We have therefore to seek a classical explanation of the observed increase in losses at centimeter wave-lengths, and the question arises what part the roughness of the metal surface may play.

It has been definitely established[4] that the effective resistance of copper wires at 3300 Mc depends markedly upon the surface treatment to which they are subjected, but the published experimental results which deal directly with the effect of surface roughness on losses are still fragmentary and incomplete. No experimental data are yet available which show quantitatively the dependence of eddy current losses on the degree of roughness of the conducting surface. In the present paper we attempt to give a theoretical discussion of this problem.

At microwave frequencies the induced currents flowing in a conductor are confined to an exceedingly thin surface layer or skin, whose thickness is measured essentially by the characteristic length

$$\delta=(2/\omega\mu g)^{\frac{1}{2}}, \tag{1}$$

where μ and g are the permeability and the conductivity of the metal in m.k.s. units and ω is the angular frequency. For copper a convenient numerical relation is

$$\delta=6.6\nu^{-\frac{1}{2}}\text{ cm},$$

where ν is the frequency in cycles per second. Thus at 10 kc, $\delta=0.066$ cm; at 1 Mc, 0.0066 cm; and at 10^4 Mc, 6.6×10^{-5} cm or 0.66 micron. Otherwise expressed, for 3-cm waves the skin thickness in copper is only 26 microinches, which is of the same order as the roughness of natural finishes such as machined or drawn surfaces. In order to determine to what extent eddy current losses may be increased by surface irregularities whose dimensions are comparable, on the average, to the skin depth, we shall calculate the theoretical magnitude of the effect in a few idealized cases.

The cases treated below are all two-dimensional; i.e., the surface roughness is assumed to consist of infinitely long parallel grooves or scratches either normal to or parallel to the direction of induced current flow. The ratio of power dissipated in a grooved surface to that dissipated in a plane surface under the same external field can be computed by straightforward if laborious numerical methods for grooves of any shape (at least for grooves transverse to the current). The relative power dissipation may be plotted against some standard quantity

[1] E. Maxwell, J. App. Phys. 18, 629 (1947).
[2] A. B. Pippard, Proc. Roy. Soc. A191, 385 (1947).
[3] G. E. H. Reuter and E. H. Sondheimer, Nature 161, 394 (1948).
[4] C. J. Milner and R. B. Clayton, J. Inst. Elec. Eng. 93, IIIa, 1409 (1946).

representing the surface roughness,[5,6] for example, the root-mean-square deviation from the mean surface, measured in units of the skin thickness. Since the curves of loss *vs.* roughness are quite similar for grooves of a few simple shapes (e.g., square, rectangular, and equilateral triangular), one may be reasonably confident that the results do not depend critically on the exact shape of the surface. This conclusion naturally breaks down for surfaces which have profiles *widely* different from those studied, or which exhibit flaws such as deep cracks or fissures. In Section IV of this paper a brief approximate treatment is given of the increase in eddy current dissipation resulting from an isolated narrow crack transverse to the current flow.

II. GROOVES TRANSVERSE TO CURRENT FLOW

Consider a semi-infinite conducting solid, as in Fig. 1a or 1b, extending from $y=f(x)$ to $y=-\infty$, where $f(x)$ is a periodic function of period d. This conductor may be part of the surface of a wave guide, coaxial pair, cavity resonator, or microwave antenna; the over-all curvature of the metal surface, if any, will be assumed negligible compared to the reciprocal of the groove width d. Let an alternating magnetic field $H_0e^{i\omega t}$ be impressed in the z direction at the surface $y=f(x)$ so that eddy currents are induced to flow parallel to the xy plane. If the conductor represents the metal surface of a wave guide, cavity resonator, or antenna, H_0 may be computed for the given mode of oscillation on the assumption that the conductivity is infinite, the alteration of the field outside the metal caused by finite conductivity being negligible to first approximation. Since the grooves are very shallow compared to the dimensions of the apparatus, the impressed field does not vary appreciably with x from point to point on the surface. Furthermore the free-space wavelength is so much greater than the depth of penetration of the induced eddy currents that we are justified in neglecting the variation of H_0 with z. Therefore, we shall treat the problem as two-dimensional and H_0 as constant.

In a material of conductivity g, dielectric constant ϵ, and permeability μ, the harmonically varying magnetic field satisfies the eddy current equation

$$\nabla^2\mathbf{H}=i\omega\mu g\mathbf{H}, \tag{2}$$

provided that the ratio $\omega\epsilon/g$ of displacement current to conduction current is negligible compared to unity. This approximation is certainly valid for copper at microwave frequencies.

If we introduce the characteristic skin thickness δ defined by (1) and recall that $\mathbf{H}$ is independent of z and has only a z component, (2) becomes

$$\partial^2H_z/\partial x^2+\partial^2H_z/\partial y^2=(2i/\delta^2)H_z, \tag{3}$$

subject to the boundary conditions $H_z=H_0$ on the surface $y=f(x)$ and $H_z\to 0$ as $|y|\to\infty$.

In the special case where $f(x)\equiv 0$, the surface of the conductor is plane; H_z is independent of x and is simply given by

$$H_z=H_0\exp[-(1+i)|y|/\delta], \tag{4}$$

this being the well-known law of penetration of all harmonically varying field quantities beneath the plane surface of a conductor. If the surface is not plane, we may still expect H_z to vanish exponentially at great depths, but in general it will depend upon both x and y.

Once we have the solution of (3), subject to the assigned boundary conditions, the power dissipated by eddy currents in a given volume V of the conducting material is evidently

$$P=(1/2g)\int_V\mathbf{J}\cdot\mathbf{J}^*dv=(1/2g)\int_V(\nabla\times\mathbf{H})\cdot(\nabla\times\mathbf{H}^*)dv, \tag{5}$$

where $\mathbf{J}$ represents the eddy current density. If we consider a cell of width d equal to the periodic distance in the x direction, unit length in the z direction, and infinite depth, the integral on the right side of (5) may be transformed, as in Appendix I, into

$$P=-(1/g\delta^2)\mathrm{Im}H_0^*\int_0^d\int_{-\infty}^{f(x)}H_zdydx, \tag{6}$$

where the integration is extended over the shaded region of Fig. 1a or 1b.

If the surface were plane, from (4) the power dissipated in a strip of unit length and of width d would be just

$$P_0=-\frac{|H_0|^2}{g\delta^2}\mathrm{Im}\int_0^\infty\int_0^d e^{-(1+i)y/\delta}dxdy=\frac{|H_0|^2d}{2g\delta}.$$

The ratio of power dissipated per unit length and

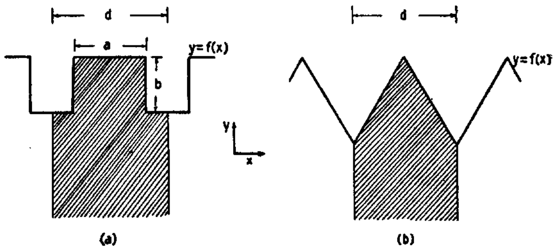

FIG. 1. Cross sections of semi-infinite conductors with rectangular and triangular surface grooves.

[5] *American Standard for Surface Roughness, Waviness, and Lay* (B46.1 American Standards Association, New York, New York, 1947).
[6] R. F. Gagg, E. R. Boynton, and James W. Owens, Industrial Standardization 19, 6 (1948).

width of grooved and smooth surfaces, respectively, is therefore

$$\frac{P}{P_0}=-\frac{2}{|H_0|^2 d\delta}\operatorname{Im}H_0^*\int_0^d\int_{-\infty}^{f(x)}H_z dy dx. \qquad (7)$$

In order to calculate the relative power dissipation numerically from (7), we have first to solve the partial differential equation (3) with its attendant boundary conditions. An approximate solution giving the values of H_z over a network of points covering the desired region may be obtained with sufficient accuracy for our purpose by the so-called "relaxation method,"[7] details of which are briefly described in Appendix II. Here we shall present merely the results.

Before carrying out numerical analysis we must specify the size and shape of the surface grooves, i.e., the form of the function $f(x)$. The simplest geometrical choices are:

(1) Rectangular:

$$f_1(x)=\begin{cases}0 \text{ for } |x|<\frac{1}{2}a,\\ -b \text{ for } \frac{1}{2}a<|x|<\frac{1}{2}d.\end{cases}$$

(2) Equilateral triangular:

$$f_2(x)=\begin{cases}-\sqrt{3}x \text{ for } 0\leq x\leq\frac{1}{2}d,\\ +\sqrt{3}x \text{ for } 0\geq x\geq-\frac{1}{2}d.\end{cases}$$

Both $f_1(x)$ and $f_2(x)$ are supposed to be periodic in x with the period d.

For the rectangular grooves the mean value of $y_1=f_1(x)$ and the r.m.s. deviation of y_1 from the mean are, respectively,

$$\bar{y}_1=-(d-a)b/d,$$
$$\Delta_1=(b/d)[a(d-a)]^{\frac{1}{2}}.$$

Similarly, for the triangular grooves

$$\bar{y}_2=-\tfrac{1}{4}\sqrt{3}d,$$
$$\Delta_2=\tfrac{1}{4}d.$$

For the actual numerical integration H_0 was assigned the arbitrary value 200 (this value obviously cancels out of Eq. (7) for the power ratio), and the values of H_z were determined to the nearest integer at the lattice points of a regular network covering the strip $0\leq x\leq\frac{1}{2}d$, account being taken of the fact that both types of grooves are symmetric with respect to the planes $x=0$ and $x=\frac{1}{2}d$.

The final values of H_z are recorded on Fig. 2 for a square groove with $a=b=2\delta$, $d=4\delta$, $\Delta_1=\delta$, on a square mesh of interval $\frac{1}{2}\delta$. Having this solution, the double integral in (7) may be evaluated by repeated application of quadrature formulas[8] analogous to Simpson's rule. For the square groove of Fig. 2 we obtain

$$\operatorname{Im}\int_0^d\int_{-\infty}^{f(x)}H_z dy dx=-2514\times\tfrac{1}{4}\delta^2.$$

Substitution into (7) gives, since $H_0=200$ and $d=4\delta$,

$$P/P_0=2514/1600=1.57.$$

It may be noted that the solution of this problem on a coarser net with mesh interval δ gave $P/P_0=1.54$, which would probably incline us to trust the above value to within one percent. This is certainly adequate, in view of the fact that we have been treating a highly idealized problem with no immediate prospects for an exact comparison of the theoretical results with experiment.

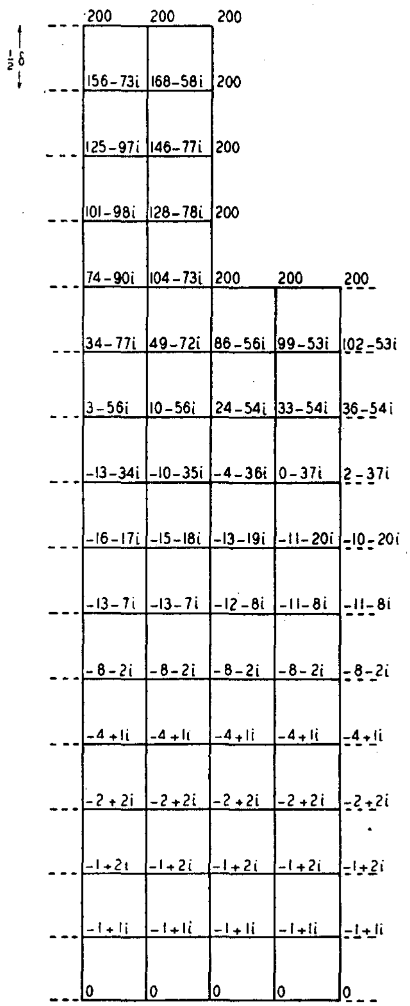

FIG. 2. Solution of the eddy current equation, $\partial^2H_z/\partial x^2+\partial^2H_z/\partial y^2=(2i/\delta^2)H_z$, for a square groove of dimensions $2\delta\times2\delta$.

[7] R. V. Southwell, *Relaxation Methods in Theoretical Physics* (Oxford University Press, London, England, 1946).

[8] *Tables of Lagrangian Interpolation Coefficients* (Columbia University Press, New York, New York, 1944), pp. xxxii–xxxiii.

Calculations exactly similar to those just outlined were carried through for three other sizes of square grooves, for which the ratio $a:b:d=1:1:2$, as well as for three sizes of rectangular grooves for which $a:b:d=3:2:4$. Finally the solution of the eddy current equation (3) was obtained for three sizes of equilateral triangular grooves, the latter calculations being carried out on a triangular-mesh net[9] with a mesh interval equal to $2\delta/3$. All of our numerical results for grooves transverse to current flow are summarized in Table I, which gives the relative power dissipation P/P_0 for different values of the ratio Δ/δ of r.m.s. roughness to skin thickness. These results are plotted in Figs. 3 and 4, where the surface profiles are also drawn to such a scale that the value of Δ is the same for each.

Qualitatively the curves of Figs. 3 and 4 require little interpretation.* It is interesting to consider a few actual numerical magnitudes for 3-cm waves in a copper pipe, where $\delta=26$ microinches. For square grooves with a *peak-to-trough* distance of 20 microinches we have $P/P_0=1.09$; for 30 microinches, 1.23; for 45 microinches, 1.49; and for 90 microinches, 1.74. It was to be anticipated that the relative power dissipation would approach the asymptotic value 2.00 for all three profiles for large values of the ratio Δ/δ, since, when the skin depth is small compared with the dimensions of the grooves, the induced currents follow the surface closely and in each case the total path length through which they flow is doubled over its value for a plane surface. We may expect in general for shallow transverse scratches, whose depth is small compared to the free-space wave-length, that in the limit when the skin depth is small compared to the average dimensions of the scratches the whole surface will be energized uniformly and the losses will merely be proportional to the total exposed area. By what factor the area of an actual metal surface exceeds the area of an ideal smooth surface of the same length and breadth is a matter for experimental rather than theoretical investigation.

III. GROOVES PARALLEL TO CURRENT FLOW

If the impressed magnetic field is parallel to the xy plane of Fig. 1, so that the induced currents all flow in the z direction parallel to the grooves, we have, in general, a much more difficult boundary value problem than that which occurred in Section II. In order to determine the electromagnetic field or the current density within the conductor, we need to know the tangential component H_t of the external magnetic field at the surface. For example, in terms of the current density J_z it is easy to show that the continuity requirement on H_t at the surface leads (at least if the trace of the surface in the xy plane consists of straight-line segments) to the boundary condition

$$\partial J_z/\partial n=(2i/\delta^2)H_t,$$

where **n**, **t**, **k** are a right-handed system of unit vectors specifying, respectively, the outward normal to the conductor, the tangent to the conductor in the xy plane, and the direction of the z axis.

Unfortunately, we do not generally know the tangential component of the surface field, and we cannot even make the convenient assumption which was possible in Section II, namely, that H_t is approximately independent of frequency. Clearly, when the frequency is zero or the conductivity is zero, the lines of H run straight across the grooves and ridges in Fig. 1, completely unaffected by the presence of the metal. Of course, no eddy currents are induced by a rigorously static field (or in a perfect dielectric). When the frequency is infinite or the conductivity is infinite the field does not penetrate at all into the metal. The boundary is a line of H, and the surface field varies markedly from point to point, being strong near sharp corners and weak at the bottoms of grooves. In the general case of finite conductivity and finite frequency, the lines of H are gradually forced out of the conductor as the frequency rises, the extent to which this occurs depending on the ratio of the dimensions of the grooves to the skin depth δ.

An exact solution of the problem of eddy current losses in grooves parallel to the current flow would require the simultaneous determination of the electromagnetic fields both outside and inside the conductor. While such an exact solution has not been carried out, it is possible to compute the external field on the assumption that the metal surface is a perfect conductor, and from this to obtain a good

TABLE I. Relative power dissipation *vs.* root-mean-square roughness for surface grooves transverse to current flow.

Δ/δ	P/P_0 Square grooves	Rectangular grooves	Triangular grooves
0	1.00	1.00	1.00
0.25	1.04	—	—
0.43	—	1.25	—
0.50	1.17	—	1.24
0.87	—	1.60	—
1.00	1.57	—	1.61
1.52	—	1.72	—
1.67	—	—	1.80
1.75	1.75	—	—

9 Reference 7, pp. 11, 20–24, *et passim*. The use of the greater mesh interval is probably justified by the fact, here pointed out, that an inherently more accurate finite difference approximation can be employed with a triangular mesh than with a square mesh.

* It is probably unnecessary to remark that slight apparent implausibilities in the relative shapes of the plotted curves, such as the more pronounced elbow near $\Delta/\delta=1.00$ of the lower curve in Fig. 4, are doubtless insignificant; they could easily be due to the uncertainty in the third figure of our values for P/P_0.

approximation to the eddy current losses when the conductivity is finite but the transverse dimensions of the grooves are large compared to the characteristic skin depth. These losses may be compared with the losses caused by grooves of similar size and shape transverse to the current, for which the power dissipation, by the remarks at the end of Section II, is merely proportional to the total surface area. In this way we may get some indication of the relative effect of the two kinds of grooves on surface losses.

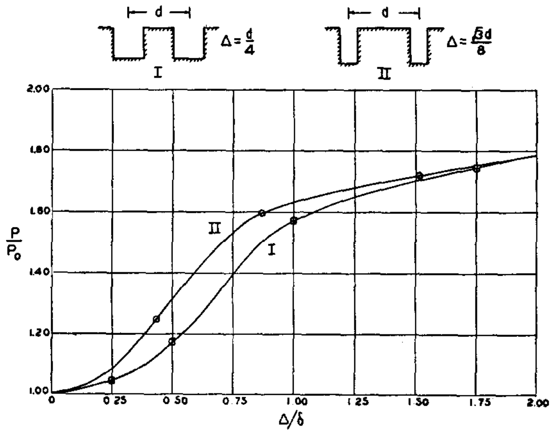

FIG. 3. Relative power dissipation P/P_0 *vs.* root-mean-square roughness Δ/δ for square and rectangular grooves transverse to current flow.

We wish, accordingly, to calculate the field distribution when an originally uniform alternating magnetic field is bounded on one side by a perfectly conducting surface having regular parallel grooves or ridges transverse to the direction of the field, the dimensions of the grooves being much smaller than the free-space wave-length corresponding to the given frequency. Advantage may be taken of the fact that, when the wave-length is sufficiently long compared with the dimensions of the region under observation, the instantaneous field configurations are indistinguishable from those given by the static solution of Laplace's equation which satisfies the same boundary conditions. To see this, we observe that the free-space propagation equation may be written in the form

$$\nabla^2\mathbf{H}=(2\pi/\lambda)^2\mathbf{H}, \tag{8}$$

where λ is the free-space wave-length. Suppose that $\mathbf{H}$ varies an appreciable fraction of its total value within a distance d; more precisely, assume that some of the second derivatives on the left side of (8) are comparable in magnitude with H/d^2. Now from (8),

$$d^2\nabla^2\mathbf{H}=4\pi^2(d/\lambda)^2\mathbf{H},$$

so that, if $d/\lambda\ll 1$, we commit a negligible error by replacing (8) with

$$\nabla^2\mathbf{H}=0. \tag{9}$$

A harmonically varying field which approximately satisfies (9) will be called "quasi-static."

A two-dimensional quasi-static magnetic field $\mathbf{H}e^{i\omega t}$ may be derived from a stream function[10] $\Psi(x, y)e^{i\omega t}$ by

$$H_x=\partial\Psi/\partial y,\quad H_y=-\partial\Psi/\partial x,\quad H_z=0, \tag{10}$$

where Ψ satisfies the two-dimensional Laplace equation

$$\partial^2\Psi/\partial x^2+\partial^2\Psi/\partial y^2=0. \tag{11}$$

On a perfectly conducting boundary the normal component of $\mathbf{H}$ vanishes and the stream function Ψ is constant. Since (11) is satisfied by either the real or the imaginary part of any analytic function W of the complex variable $\zeta=x+iy$, our problem reduces to finding a suitable complex stream function,

$$W=\Phi+i\Psi, \tag{12}$$

whose imaginary part is constant on conducting boundaries. For a grooved or ridged boundary whose trace on the ζ-plane consists entirely of straight-line segments, W may be found by a Schwarz transformation.[10] The transformation giving the rectangular grooves of Fig. 1a is derived in terms of elliptic functions in Appendix III.

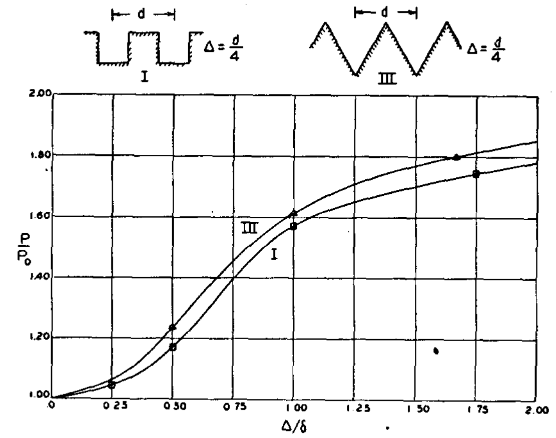

FIG. 4. Relative power dissipation P/P_0 *vs.* root-mean-square roughness Δ/δ for square and equilateral triangular grooves transverse to current flow.

The power dissipated by eddy currents in a surface S of conductivity g, whose radius of curvature is large compared to the skin depth δ, is given approximately by[11]

$$P=(1/2g\delta)\int_S|H_t|^2ds, \tag{13}$$

[10] S. A. Schelkunoff, *Electromagnetic Waves* (D. van Nostrand Company, Inc., New York, New York, 1943), pp. 174, 179–180, 184 ff.
[11] Reference 10, p. 320.

where H_t is the amplitude of the tangential magnetic field impressed at the conducting surface. For the grooved surface under consideration, if the skin depth is very small compared to the transverse dimensions of a typical groove, we may obtain a good approximation to the dissipated power by using in (13) the value of H_t which has been calculated on the assumption of a perfectly conducting surface. Of course (13) does not apply at a corner where the radius of curvature is zero; but if the size of the grooves is large compared to the skin depth only a small fraction of the total power will be dissipated in the immediate vicinity of sharp corners.

To the approximation considered, the integral in (13) is evaluated for the case of rectangular grooves in Appendix III. The ratio of eddy current power spent in a groove of width d to the power spent in a plane strip of width d and equal length is found to be

$$\frac{P}{P_0}=\frac{2K'}{\pi}\left[\frac{k'^2\mathrm{sn}(h,k')\mathrm{cn}(h,k')}{\mathrm{dn}(h,k')}\right]+1-\frac{2b}{d}, \quad (14)$$

where the quantities h and k' are related to the dimensions a, b, d of Fig. 1a by the following pair of equations:

$$h=(a/d)K'+(2b/d)K, \quad (15)$$

$$2Z(h,k')-\frac{2k'^2\mathrm{sn}(h,k')\mathrm{cn}(h,k')}{\mathrm{dn}(h,k')}+\frac{\pi(2b/d)}{K'}=0. \quad (16)$$

In these equations K and K' are the complete elliptic integrals of the first kind to the complementary moduli k and k', respectively, and $Z(h, k')$ is the zeta-function of Jacobi.[12] Thus we see that, whereas the limiting value of P/P_0 for grooves transverse to the current and large compared to the skin depth is just equal to the relative increase in total surface area, the limiting value for rectangular grooves parallel to the current is a function of the root of a pair of complicated transcendental equations involving the dimensions of the grooves.

Numerical results have been obtained only for grooves of square cross section, i.e., grooves with

$$a=b=\tfrac{1}{2}d.$$

In this case Eqs. (15) and (16) become

$$h=K+K'/2,$$

$$Z(h,k')-\frac{k'^2\mathrm{sn}(h,k')\mathrm{cn}(h,k')}{\mathrm{dn}(h,k')}+\frac{\pi}{2K'}=0.$$

[12] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, London, England, 1940), fourth edition, p. 518.

These equations were solved for k by a trial and error process using a table of elliptic functions;[13] the quantities of interest were found to be

$$k=\sin 3°16'=0.05698,\quad k'=\sin 86°44'=0.99838,$$
$$K=1.5721,\quad K'=4.2545,\quad h=3.6993.$$

Substitution of these numerical values into (14) gives for the limiting ratio of eddy current losses resulting from square grooves parallel to the current:

$$(P/P_0)_{\parallel}=1.360. \quad (17)$$

Since the corresponding ratio for square grooves transverse to the current is

$$(P/P_0)_{\perp}=2.000, \quad (18)$$

the increase in power dissipation due to large "parallel" grooves is only about one-third as much as the increase due to large "perpendicular" grooves in this case. Whether the curves representing $(P/P_0)_{\parallel}$ *vs.* Δ/δ would be completely similar in shape to the curves of Figs. 3 and 4 for $(P/P_0)_{\perp}$ of course we cannot be sure, but from the comparative asymptotic values (17) and (18) it seems reasonable to believe that, insofar as anomalous eddy current losses may be attributable to surface grooves and scratches, transverse grooves have a considerably greater adverse effect than grooves parallel to the current.

IV. ISOLATED DEEP CRACK TRANSVERSE TO CURRENT

In addition to shallow scratches it is quite likely that a metal surface which has been strained in the process of working will exhibit occasional narrow cracks and fissures whose depth may be several times the skin thickness, though still small compared with the free-space wave-length corresponding to the given frequency. An approximation to the eddy current power dissipated in such a crack transverse to the current may be obtained by regarding it as a section of parallel-plane transmission line.

Consider an infinitely long rectangular groove of width b and length l (Fig. 5) in a semi-infinite conducting medium of electrical constants ϵ, μ, g. Neglecting edge effects near the opening, unit length of this groove is equivalent to a transmission line[14] for transverse magnetic waves (H perpendicular to plane of paper) whose distributed series impedance and shunt admittance per unit distance

[13] E. P. Adams and R. L. Hippisley, *Smithsonian Mathematical Formulae and Tables of Elliptic Functions* (Smithsonian Institution, Washington, D. C., 1922), pp. 260–309. Since we have used linear interpolation uncritically with these tables, the numerical values in this section are probably given to one more figure than is significant.

[14] Reference 10, pp. 484 ff.

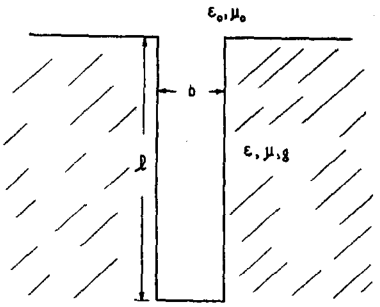

FIG. 5. Cross section of isolated rectangular groove in a conducting surface.

are, respectively,

$$Z=2\eta+i\omega L=2(1+i)/g\delta+i\omega\mu_0 b, \tag{19}$$

$$Y=i\omega C=i\omega\epsilon_0/b. \tag{20}$$

Here ϵ_0, μ_0 are the dielectric constant and permeability of free space, δ is the skin thickness defined by (1), and

$$\eta=(i\omega\mu/g)^{\frac{1}{2}}=(1+i)/g\delta \tag{21}$$

is the intrinsic impedance of the conducting material at the given angular frequency ω.

From familiar transmission line theory[15] we know that if the line is terminated in an impedance

$$Z_l=\eta b=b(1+i)/g\delta,$$

the input impedance will be

$$Z_i=K\frac{Z_l\,\mathrm{ch}\Gamma l+K\,\mathrm{sh}\Gamma l}{K\,\mathrm{ch}\Gamma l+Z_l\,\mathrm{sh}\Gamma l}, \tag{22}$$

where

$$\Gamma=(YZ)^{\frac{1}{2}},\quad K=(Z/Y)^{\frac{1}{2}}. \tag{23}$$

Thus the ratio of the power dissipated by a current I flowing across the groove to the power dissipated in the absence of the groove by the same current flowing across a flat strip of width b is

$$\frac{P}{P_0}=\frac{\mathrm{Re}\frac{1}{2}Z_i|I|^2}{\mathrm{Re}\frac{1}{2}\eta b|I|^2}=\frac{\mathrm{Re}Z_i}{(b/g\delta)}. \tag{24}$$

In the special case where the width of the groove is comparable to or greater than the skin thickness and the depth is much less than the free-space wave-length, i.e., $b/\delta\approx 1$ and $l/\lambda\ll 1$, we have from (19), (20), and (23), on writing $\lambda=(2\pi/\omega)(\mu_0\epsilon_0)^{-\frac{1}{2}}$, the relation

$$|\Gamma l|=(2\pi l/\lambda)|(1+\delta/b)-i\delta/b|^{\frac{1}{2}}\approx(2\pi l/\lambda)\ll 1.$$

Hence $\mathrm{sh}\Gamma l\approx\Gamma l$ and $\mathrm{ch}\Gamma l\approx 1$, and substituting these approximations into (22) we get, after some manipulation,

$$Z_i\approx\frac{(1/g)[(b+2l)/\delta+i(b/\delta+2l/\delta+2lb/\delta^2)]}{1+\frac{1}{2}(i-1)(2\pi/\lambda)^2 l\delta}.$$

The second term in the denominator is negligible compared with unity under the given assumptions. Hence, from (24) the ratio of power dissipated in the groove to power dissipated on the plane surface is simply

$$P/P_0=(b+2l)/b. \tag{25}$$

Thus the increase in eddy current loss is proportional to the additional surface area introduced by the walls of the groove, provided $b\approx\delta$.

For a very narrow crack it is to be expected that the shunt capacitance between the walls will become the dominant effect and will ultimately short-circuit the crack. To investigate this we assume now that $b/\delta\ll 1$ and obtain the following approximations:

$$\begin{aligned}
Z_l&=(b/\delta)(1+i)/g,\\
\Gamma l&\approx(2\pi l/\lambda)(i-1)^{\frac{1}{2}}(b/\delta)^{-\frac{1}{2}},\\
K&\approx -i(\lambda/\pi g\delta)(i-1)^{\frac{1}{2}}(b/\delta)^{\frac{1}{2}},\\
\mathrm{ch}\Gamma l&\approx\tfrac{1}{2}\exp[(2\pi l/\lambda)(i-1)^{\frac{1}{2}}(b/\delta)^{-\frac{1}{2}}]\approx\mathrm{sh}\Gamma l,\\
Z_i&\approx K\approx(1.099-0.455i)(\lambda/\pi g\delta)(b/\delta)^{\frac{1}{2}}.
\end{aligned}$$

The input impedance of a groove of fixed depth thus vanishes like $(b/\delta)^{\frac{1}{2}}$ when the ratio $b/\delta\to 0$, but not so fast as the impedance of a plane strip of width b, which vanishes directly as b/δ. If b/δ is so small that the last approximation for Z_i holds, the ratio of powers is given by (24) to be

$$P/P_0=1.099(\lambda/\pi\delta)(b/\delta)^{-\frac{1}{2}}. \tag{26}$$

V. CONCLUSIONS

The effect of shallow parallel scratches on the surface of a conductor is to increase eddy current losses by an appreciable amount if the dimensions of the scratches are comparable on the average to the skin depth. If the scratches are transverse to the direction of induced current flow the increase in loss may be anywhere between zero and approximately 100 percent, depending on the ratio of r.m.s. surface roughness to skin depth. This increase does not depend critically on the exact shape of the surface profile, so long as the scratches are of approximately equal width and depth.

The adverse effect on eddy current losses of scratches parallel to the current flow is considerably less than the adverse effect caused by transverse scratches of similar size. Our analysis suggests that the increase in relative power dissipation resulting from "parallel" grooves or scratches may be only slightly more than one-third as great as the increase in the transverse case.

The eddy current dissipation in an isolated narrow crack transverse to the current is approxi-

[15] Reference 10, Chapter VII.

mately proportional to the total surface area of the walls, provided that the depth of the crack is much less than the free-space wave-length corresponding to the given frequency and the width is comparable to or greater than the skin thickness. For an extremely narrow crack the shunt capacitance between the walls becomes important, and the losses are ultimately proportional to the square root of the ratio of width to skin thickness.

ACKNOWLEDGMENTS

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APPENDIX I

Transformation of the integral

$$\int_V (\nabla\times\mathbf{H})\cdot(\nabla\times\mathbf{H}^*)dv.$$

In the integral

$$2gP=\int_V (\nabla\times\mathbf{H})\cdot(\nabla\times\mathbf{H}^*)dv, \tag{A1}$$

write

$$\nabla\times\mathbf{H}=\nabla\times H_z\mathbf{k}=\nabla H_z\times\mathbf{k},$$

where $\mathbf{k}$ is the unit vector in the z direction. Then,

$$2gP=\int_V (\nabla H_z)\cdot(\nabla H_z^*)dv,$$

since ∇H_z is normal to $\mathbf{k}$. On applying Green's theorem,

$$\int_V \nabla U\cdot\nabla W dv=\int_S W(\partial U/\partial n)ds-\int_V W\nabla^2 U dv,$$

first with $U=H_z$ and $W=H_z^*$, then with $U=H_z^*$ and $W=H_z$, we obtain

$$2gP=\tfrac{1}{2}\left[\int_S (H_z^*(\partial H_z/\partial n)+H_z(\partial H_z^*/\partial n))ds\right.$$
$$\left.-\int_V (H_z^*\nabla^2 H_z+H_z\nabla^2 H_z^*)dv\right], \tag{A2}$$

where n is the outward normal to the surface S bounding the volume V. In virtue of (3) and its complex conjugate equation,

$$\nabla^2 H_z^*=-(2i/\delta^2)H_z^*,$$

the volume integral in (A2) vanishes. If the integration is extended over a cell of unit length and of width d, the surface integrals over the ends $z=$ constant vanish because $\partial H_z/\partial z=0$, and the contributions of the side surfaces $x=x_0$ and $x=x_0+d$ cancel each other on account of periodicity. The fields vanish exponentially as $|y|\rightarrow\infty$, so we are left with only the integral over the external surface S_1 where $H_z=H_0=$ constant. Hence

$$2gP=\mathrm{Re}\int_{S_1} H_z^*(\partial H_z/\partial n)ds=\mathrm{Re}H_0^*\int_{S_1}(\partial H_z/\partial n)ds.$$

The last integral may be re-extended over the entire surface S of the volume under consideration since, as before, the net contribution of the sides and the surface at infinity will be zero. Once more applying Green's theorem, this time with $U=H_z$ and $W=1$, and then Eq. (3), we finally get

$$2gP=\mathrm{Re}H_0^*\int_S(\partial H_z/\partial n)ds=\mathrm{Re}H_0^*\int_V \nabla^2 H_z dv$$
$$=\mathrm{Re}H_0^*\int_V (2i/\delta^2)H_z dv$$
$$=-(2/\delta^2)\mathrm{Im}H_0^*\int_0^d\int_{-\infty}^{f(x)} H_z dy dx,$$

which is Eq. (6) of Section II.

APPENDIX II

Relaxation method applied to the eddy current equation. We seek a numerical solution of the equation

$$\partial^2 w/\partial x^2+\partial^2 w/\partial y^2=(2i/\delta^2)w, \tag{A3}$$

which takes the constant value $w=W_0$ on the boundary $y=f(x)$, is symmetric with respect to the lines $x=0$ and $x=\frac{1}{2}d$, and vanishes as $y\rightarrow-\infty$.

The first step is to replace the partial derivatives in (A3) by finite differences. Let w_0 be the value of w at an interior point of a square-mesh net such as that of Fig. 2, and let w_1, w_2, w_3, w_4 be the surrounding values, as in Fig. 6. Then it may be shown[16] that the finite-difference approximation to (A3) is

$$\tfrac{1}{4}\sum_{n=1}^{4} w_n-w_0[1+(ih^2/2\delta^2)]=0, \tag{A4}$$

up to terms of order h^4, where h is the mesh interval.

The relaxation method of solving (A4) consists in the following procedure: Having assumed over the given net a solution, however crude, which

[16] Reference 7, p. 20.

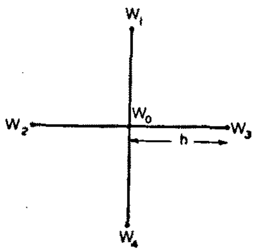

FIG. 6. Typical point of a "relaxation net."

satisfies the boundary conditions, we define at each interior point a *residual* R proportional to the left side of (A4). By altering the value of w_0 at a given point we can reduce the value of the residual at that point, at the expense of simultaneous changes in the residuals at the immediately adjacent points; successive corrections at points where the residuals are largest gradually bring us closer to the desired solution. Theoretically there exists a set of w's for which every residual is zero; in practice, however, we adopt some limit of tolerance and regard the solution as complete when all residuals are reduced below this value.

As an example, to obtain the solution of Fig. 2 we take a mesh interval $h=\frac{1}{2}\delta$ in (A4). Multiplying the left side of (A4) by 8 to clear of fractions and writing $w=u+iv$, we define the residual to be

$$R=(2\sum_{n=1}^{4}u_n-8u_0+v_0)+i(2\sum_{n=1}^{4}v_n-u_0-8v_0). \quad \text{(A5)}$$

We then assume, using as an aid to guessing any physical insight or *a priori* knowledge of the nature of the solution that we may possess, a set of values of w and compute the initial residuals by (A5), noting that if w_0 lies on one of the lateral boundaries of symmetry, then $w_2=w_3$. It is convenient to decide in advance the number of decimals which will be wanted in the solution and then to multiply the boundary value W_0 by such a factor that the actual calculations may be done in integers. Thereafter the values of w and the residuals are altered according to the following scheme, derived from (A5):

Δw_0	ΔR_0	$\Delta R_{1,2,3,4}$
$+1$	$-8-i$	$+2$
$+1$	$+1-8i$	$+2i$

If w_2 (or w_3) lies on a boundary of symmetry, then the corresponding ΔR is $+4$ or $+4i$, respectively, because the change in w_0 is reflected on the other side of the boundary.

The degree to which liquidation of residuals can ultimately be carried depends on the minimum correction we are willing to make in w; for example, if we round off w to the nearest integer we should accept residuals whose real and imaginary parts separately do not exceed 4 units, if they are defined by (A5). This limit of tolerance was imposed in obtaining the solution of Fig. 2. When a solution is deemed acceptable, one checks the work by recomputing all residuals directly from the final values of w.

An empirical check on the accuracy of our results for the square groove of dimensions $2\delta\times2\delta$ was provided by carrying out the complete solution on two different nets. The mesh intervals employed were $h=\delta$ and $h=\frac{1}{2}\delta$; the corresponding values of the power ratio P/P_0 were 1.54 for the coarse net and 1.57 for the fine net. A mesh interval of $\frac{1}{2}\delta$ was then used for all the rectangular grooves except the square one of dimensions $\frac{1}{2}\delta\times\frac{1}{2}\delta$, in which case the interval was $\frac{1}{4}\delta$.

For the finite-difference approximation analogous to (A4) which may be used with a triangular-mesh net, reference should be had to a text[9] on relaxation methods.

APPENDIX III

Conformal representation of a boundary with rectangular grooves. Starting with the complex stream function

$$W=H_0\zeta, \quad \text{(A6)}$$

which represents a uniform field H_0 parallel to the x axis, we subject the ζ-plane to the following conformal transformations (Fig. 7):

$\zeta_1=\zeta_1(\zeta)$ takes a semi-infinite vertical strip of width $2a$ in the ζ-plane into the entire upper half of the ζ_1-plane.

$\zeta_2=\zeta_2(\zeta_1)$ takes the upper half of the ζ_1-plane into a semi-infinite vertical strip of width $2a_2$ in the ζ_2-plane, with a symmetrical rectangular notch of height b_2 and width $2c_2$ in the lower boundary.

Let it be assumed that the following correspondences exist between pairs of points in Fig. 7, where for reasons of symmetry the notation differs somewhat from that employed in Fig. 1a and in the main body of this paper:

$$0\rightleftarrows 0_1\rightleftarrows 0_2, \quad \text{(A7)}$$

$$c\rightleftarrows c_1\rightleftarrows c_2, \quad \text{(A8)}$$

$$b\rightleftarrows b_1\rightleftarrows c_2-ib_2, \quad \text{(A9)}$$

$$a\rightleftarrows a_1\rightleftarrows a_2-ib_2. \quad \text{(A10)}$$

Then the required transformations are, in differential form:

$$d\zeta/d\zeta_1=C_1(\zeta_1^2-a_1^2)^{-\frac{1}{2}}, \quad \text{(A11)}$$

$$d\zeta_2/d\zeta_1=C_2(\zeta_1^2-a_1^2)^{-\frac{1}{2}}(\zeta_1^2-b_1^2)^{-\frac{1}{2}}(\zeta_1^2-c_1^2)^{\frac{1}{2}}, \quad \text{(A12)}$$

and hence

$$d\zeta_2/d\zeta=C(\zeta_1^2-b_1^2)^{-\frac{1}{2}}(\zeta_1^2-c_1^2)^{\frac{1}{2}}, \quad \text{(A13)}$$

where C_1, C_2, and $C=C_2/C_1$ are as yet arbitrary constants. From (A11) and (A13) and the left members of (A7)–(A10) we easily obtain

$$\zeta_1 = a_1 \sin(\pi\zeta/2a)$$

and

$$\zeta_2 = \int_0^{\zeta} \left[\frac{\sin^2(\pi\zeta/2a) - \sin^2(\pi c/2a)}{\sin^2(\pi\zeta/2a) - \sin^2(\pi b/2a)} \right]^{\frac{1}{2}} d\zeta, \quad \text{(A14)}$$

the constant C in (A13) having been put equal to unity so that we shall have $d\zeta_2/d\zeta \to 1$ when $\mathrm{Im}\,\zeta$ is large and positive. The constants a, b, and c have ultimately to be evaluated in terms of a_2, b_2, and c_2, which specify the dimensions of the groove; evidently $a = a_2$ if $C = 1$.

To simplify writing put

$$\theta = \pi\zeta/2a, \quad \beta = \pi b/2a, \quad \gamma = \pi c/2a. \quad \text{(A15)}$$

The substitutions

$$t = \tan\theta/\tan\gamma, \quad k = \tan\gamma/\tan\beta \quad \text{(A16)}$$

give

$$\zeta_2 = (2a/\pi) \int_0^{\theta} [(\sin^2\theta - \sin^2\gamma)/(\sin^2\theta - \sin^2\beta)]^{\frac{1}{2}} d\theta$$

$$= \frac{2a}{\pi} \frac{\sin^2\gamma}{\sin\beta\cos\gamma} \left[\int_0^t \frac{dt}{[(1-t^2)(1-k^2t^2)]^{\frac{1}{2}}} \right.$$

$$\left. -\sec^2\gamma \int_0^t \frac{t^2 dt}{(1+t^2\tan^2\gamma)[(1-t^2)(1-k^2t^2)]^{\frac{1}{2}}} \right]. \quad \text{(A17)}$$

Let

$$t = \mathrm{sn}(u, k), \quad u = \mathrm{sn}^{-1}(t, k), \\ du = [(1-t^2)(1-k^2t^2)]^{-\frac{1}{2}} dt. \quad \text{(A18)}$$

Then

$$\zeta_2 = \frac{2a}{\pi} \frac{\sin^2\gamma}{\sin\beta\cos\gamma} \left[u - \sec^2\gamma \int_0^u \frac{\mathrm{sn}^2 u du}{1 + \tan^2\gamma\, \mathrm{sn}^2 u} \right]. \quad \text{(A19)}$$

The last integral is an elliptic integral of the third kind,[17] which will be in the standard form if we write

$$\tan^2\gamma = -k^2 \mathrm{sn}^2(ih, k), \quad \text{(A20)}$$

i.e., recalling (A16),

$$\mathrm{sn}(ih, k) = i\tan\beta, \quad \mathrm{cn}(ih, k) = \sec\beta, \\ \mathrm{dn}(ih, k) = \sec\gamma. \quad \text{(A21)}$$

Jacobi's imaginary transformation[18] gives

$$\mathrm{sn}(h, k') = \sin\beta, \quad \mathrm{cn}(h, k') = \cos\beta, \\ \mathrm{dn}(h, k') = \cos\beta\sec\gamma, \quad \text{(A22)}$$

where

$$k^2 + k'^2 = 1.$$

From (A22), since $0 < \beta < \pi/2$, h is real and

$$0 < h = \mathrm{am}^{-1}(\beta, k') < K'.$$

[17] Reference 12, pp. 522–523.
[18] Reference 12, pp. 505–506.

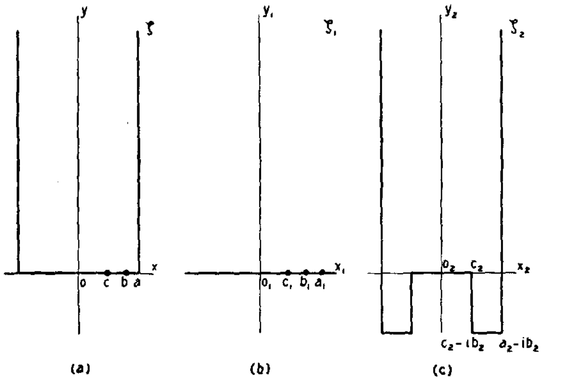

FIG. 7. Conformal transformation of a plane boundary into a boundary with regular rectangular grooves.

Now

$$\int_0^u \frac{\mathrm{sn}^2 u du}{1 - k^2\mathrm{sn}^2(ih)\mathrm{sn}^2 u}$$

$$= \frac{1}{k^2 \mathrm{sn}(ih)\mathrm{cn}(ih)\mathrm{dn}(ih)} \times \left[\tfrac{1}{2} \log \frac{\Theta(u-ih)}{\Theta(u-ih)} + uZ(ih) \right], \quad \text{(A23)}$$

where the modulus of the elliptic functions is understood to be k unless otherwise indicated. It will be convenient to use the relation[19]

$$Z(ih, k) = i \left[\frac{\mathrm{dn}(h, k')\mathrm{sn}(h, k')}{\mathrm{cn}(h, k')} - Z(h, k') - \frac{\pi h}{2KK'} \right]. \quad \text{(A24)}$$

Substituting (A23) and (A24) into (A19) and employing (A21) and (A22) to replace the trigonometric functions of β and γ with functions of h and k, we obtain after considerable reduction

$$\zeta_2 = \frac{a}{\pi} \left[Au + i \log \frac{\Theta(u-ih)}{\Theta(u+ih)} \right], \quad \text{(A25)}$$

where

$$A = 2Z(h, k') + \frac{\pi h}{KK'} - \frac{2k'^2 \mathrm{sn}(h, k')\mathrm{cn}(h, k')}{\mathrm{dn}(h, k')}. \quad \text{(A26)}$$

As ζ varies from 0 to a, t varies from 0 to ∞, and u from 0 to K to $K+iK'$ to iK'. In tracing the variation of ζ_2 we must be careful to stay on the same branch of the logarithmic function. If we do so, we find the correspondences in the following

[19] Reference 12, p. 519.

table:

ζ	t	u	ζ_2
0	0	0	0
c	1	K	aAK/π
b	$1/k$	$K+iK'$	$aA[K+iK']/\pi - aih/K$
a	∞	iK'	$aAiK'/\pi - aih/K + a$

If we now introduce the constants a_2, b_2, c_2 which specify the dimensions of a typical groove in Fig. 7c, we see from (A8)–(A10) that the following relations must hold:

$$aAK/\pi = c_2, \tag{A27}$$

$$ah/K - aAK'/\pi = b_2, \tag{A28}$$

$$a = a_2. \tag{A29}$$

The scale factors a and a_2 are equal, as previously noted; h and k depend on the ratios c_2/a_2 and b_2/a_2. Elimination of A from (A27) and (A28) gives

$$h = (c_2/a_2)K' + (b_2/a_2)K; \tag{A30}$$

this value of h may be substituted into the expression for A given by (A29). Thus, employing (A26) with (A27),

$$2Z(h, k') - \frac{2k'^2 \operatorname{sn}(h, k') \operatorname{cn}(h, k')}{\operatorname{dn}(h, k')} + \frac{\pi(b_2/a_2)}{K'} = 0. \tag{A31}$$

Comparing Fig. 7c with Fig. 1a, we see that the quantities a_2, b_2, c_2 correspond to $\frac{1}{2}d$, b, $\frac{1}{2}a$, respectively, so that (A30) and (A31) are equivalent to (15) and (16) of Section III.

If Ψ and W are defined by Eqs. (10) and (12) of Section III, it follows from the Cauchy-Riemann relations[10] that the magnitude of the field represented by W at any point of the ζ_2-plane is just

$$H = |\partial W/\partial \zeta_2| = |H_0|\,|\partial\zeta/\partial\zeta_2|.$$

Employing Eq. (13) to calculate the relative power dissipation in the grooved surface, we have

$$\frac{P}{P_0} = \left[\int_0^{a_2} |H_0|^2 |d\zeta/d\zeta_2|^2 |d\zeta_2|\right] \Big/ \left[\int_0^{a} |H_0|^2 |d\zeta|\right]$$

$$= \frac{1}{a}\int_0^a \left|\frac{\sin^2(\pi\zeta/2a) - \sin^2(\pi b/2a)}{\sin^2(\pi\zeta/2a) - \sin^2(\pi c/2a)}\right|^{\frac{1}{2}} |d\zeta|, \tag{A32}$$

if we make use of (A14). We introduce again the substitutions (A15) and (A16) and obtain

$$\frac{P}{P_0} = \frac{2\sin\beta}{\pi\cos\gamma}\int_0^\infty \left|1 - \frac{t^2\tan^2\gamma\csc^2\beta}{1+t^2\tan^2\gamma}\right| \times \frac{dt}{|(1-t^2)(1-k^2t^2)|^{\frac{1}{2}}}. \tag{A33}$$

In order to make use of our previous work, we shall find it convenient to have the abbreviations

$$U_{p,q} \equiv \int_p^q \frac{dt}{|(1-t^2)(1-k^2t^2)|^{\frac{1}{2}}},$$

$$V_{p,q} \equiv \int_p^q \frac{t^2 dt}{(1+t^2\tan^2\gamma)|(1-t^2)(1-k^2t^2)|^{\frac{1}{2}}}.$$

It follows from (A17) and from the table below (A26) that

$$U_{0,1} = K, \quad V_{0,1} = \left[\cos^2\gamma - \frac{A\sin\beta\cos\gamma}{2\tan^2\gamma}\right]K;$$

$$U_{1,1/k} = K',$$

$$V_{1,1/k} = \left[\cos^2\gamma - \frac{A\sin\beta\cos\gamma}{2\tan^2\gamma}\right]K' + \frac{\pi h\sin\beta\cos\gamma}{2K\tan^2\gamma};$$

$$U_{1/k,\infty} = K,$$

$$V_{1/k,\infty} = \left[\cos^2\gamma - \frac{A\sin\beta\cos\gamma}{2\tan^2\gamma}\right]K + \frac{\pi\sin\beta\cos\gamma}{2\tan^2\gamma}.$$

Hence from (A33)

$$\frac{P}{P_0} = \frac{2\sin\beta}{\pi\cos\gamma}\left[U_{0,1/k} - \frac{\tan^2\gamma}{\sin^2\beta}V_{0,1/k} - U_{1/k,\infty} + \frac{\tan^2\gamma}{\sin^2\beta}V_{1/k,\infty}\right]$$

$$= \frac{2K'}{\pi}\left[\frac{\sin^2\beta - \sin^2\gamma}{\sin\beta\cos\gamma}\right] + \frac{AK'}{\pi} - \frac{h}{K} + 1.$$

If we substitute for A from (A27) and for h from (A30), the expression for P/P_0 simplifies to

$$\frac{P}{P_0} = \frac{2K'}{\pi}\left[\frac{\sin^2\beta - \sin^2\gamma}{\sin\beta\cos\gamma}\right] + 1 - \frac{b_2}{a_2},$$

which is equivalent, by (A16) and (A22), to Eq. (14) of Section III.